UNIVERSITY OF Southampton

Hamiltonian Fluid Dynamics & Irrotational Binary Inspiral

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work in progress in collaboration with: John Friedman, Masaru Shibata, Niclas Moldenhauer, David Hilditch, Sebastiano Bernuzzi, Koutarou Kyutoku, Bernd Brüegmann

Introduction

- Gravitational waves from neutron-star and black-hole binaries carry valuable information on their physical properties and probe physics inaccessible to the laboratory.
- Although development of **black-hole** gravitational wave templates in the past decade has been revolutionary, the corresponding work for double **neutron-star** systems has lagged.
- Recent progress by groups in Kyoto (SACRA), Caltech-Cornell-CITA-AEI (SpEC), Frankfurt (Whisky), Jena (BAM), Illinois, etc.
- The **Valencia** scheme has been a workhorse for hydro in numerical relativity…

$$\nabla_\alpha(\rho u^\alpha) = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}\,\rho u^\alpha) = 0$$

$$\nabla_\beta T_\alpha{}^\beta = \frac{1}{\sqrt{-g}}\partial_\beta(\sqrt{-g}\,T_\alpha{}^\beta) - \Gamma^\gamma_{\alpha\beta}T_\gamma{}^\beta = 0$$

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- The **Valencia** scheme has been a workhorse for hydro in numerical relativity, but considering alternative hydrodynamic schemes can lead to further progress…
- **Hamiltonian** methods have been used in all areas of physics but have seen little use in hydrodynamics

Introduction

- Constructing a Hamiltonian requires a variational principle
- Carter and Lichnerowicz have described barotropic fluid motion via classical variational principles as **conformally geodesic**

$$\delta\int_a^b h\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}d\tau = 0 \qquad h = 1 + \int_0^\rho \frac{dp}{\rho}$$

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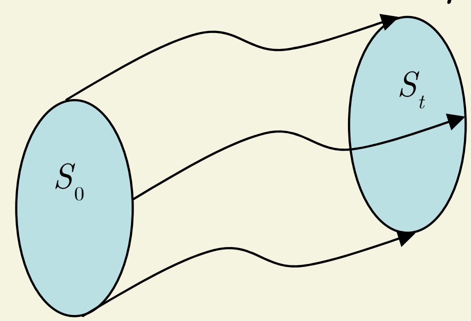
- Moreover, Kelvin's circulation theorem

$$\frac{d}{d\tau}\oint_{c_\tau} hu_\alpha dx^\alpha = 0$$



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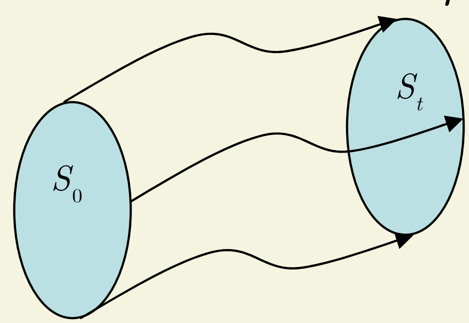

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- Applied to numerical relativity, these concepts lead to novel **Hamiltonian** or **Hamilton-Jacobi** schemes for evolving relativistic fluid flows, applicable to binary neutron star inspiral.

Carter-Lichnerowicz variational principles for barotropic flows

- Carter's Lagrangian: $$\mathcal{L} = \frac{h}{2} g_{\alpha\beta} u^{\alpha} u^{\beta} - \frac{h}{2} = -h \text{ (on shell)}$$
- Canonical momentum: $$p_{\alpha} = \frac{\partial \mathcal{L}}{\partial u^{\alpha}} = h u_{\alpha}; \qquad u^{\alpha} = \frac{dx^{\alpha}}{d\tau}$$
- Carter's superHamiltonian: $$\mathcal{H} = p_{\alpha} u^{\alpha} - \mathcal{L} = \frac{1}{2h} g^{\alpha\beta} p_{\alpha} p_{\beta} + \frac{h}{2} = 0$$

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- Euler equation in Carter-Lichnerowicz form:

$$\frac{dp_{\alpha}}{d\tau} - \frac{\partial \mathcal{L}}{\partial x^{\alpha}} = \pounds_{u} p_{\alpha} - \nabla_{\alpha} \mathcal{L} = 0 \quad \text{(Euler-Lagrange)}$$

$$\frac{dp_{\alpha}}{d\tau} + \frac{\partial \mathcal{H}}{\partial x^{\alpha}} = u^{\beta} (\nabla_{\beta} p_{\alpha} - \nabla_{\alpha} p_{\beta}) + \nabla_{\alpha} \mathcal{H} = 0 \quad \text{(Hamilton)}$$

Constrained Hamiltonian approach

$$\delta\int_a^b h\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}}dt = \delta\int_a^b \alpha h\sqrt{1-\gamma_{ab}\nu^a\nu^b}dt = 0$$

$\nu^a = \alpha^{-1}(v^a + \beta^a)$ fluid velocity measured by normal observers

$v^a = dx^a / dt$ fluid velocity measured in local coordinates

$p_a = \dfrac{\partial L}{\partial v^a} = h\dfrac{\nu_a}{\sqrt{1-\nu^2}} = hu_a$ canonical momentum of a fluid element

Constrained Hamiltonian approach

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Constrained Hamiltonian: $H = p_a v^a - L = -p_a \beta^a + \alpha \sqrt{h^2 + \gamma^{ab} p_a p_b} = -h u_t$

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"*You've got the action, you've got the motion*" – Dire Straights

Constrained Hamiltonian approach

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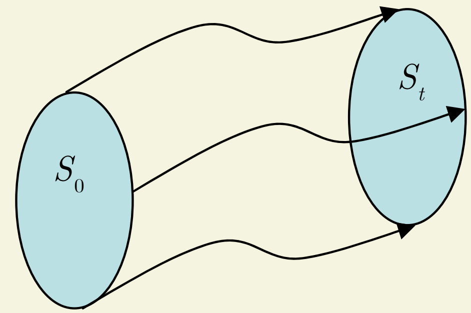
Euler-Lagrange equation: $\dfrac{dp_a}{dt} - \dfrac{\partial L}{\partial x^a} = (\partial_t + \pounds_v)p_a - \nabla_a L = 0$

Hamilton equation: $\dfrac{dp_a}{dt} + \dfrac{\partial H}{\partial x^a} = \partial_t p_a + v^b(\nabla_b p_a - \nabla_a p_b) + \nabla_a H = 0$

(independent of gravity theory)

Conservation of circulation

Euler-Lagrange equation: $(\partial_t + \pounds_v)p_a = \nabla_a L \Rightarrow (\partial_t + \pounds_v)\omega_{ab} = 0$



Vorticity 2-form: $\omega_{ab} = \nabla_a p_b - \nabla_b p_a$

Kelvin's theorem: $\dfrac{d}{dt}\oint_{\mathcal{C}_t} p_a dx^a = \dfrac{d}{dt}\int_{\mathcal{S}_t} \omega_{ab}\, dx^a \wedge dx^b = \int_{\mathcal{S}_0} (\partial_t + \pounds_v)\omega_{ab}\, dx^a \wedge dx^b = 0$

- The most interesting feature of Kelvin's theorem is that, since its derivation did not depend on the metric, it is **exact in time-dependent spacetimes, with gravitational waves carrying energy and angular momentum away from a system**. In particular, oscillating stars and radiating binaries, if modeled as barotropic fluids with no viscosity or dissipation other than gravitational radiation **exactly** conserve circulation

- **Corollary**: flows initially irrotational remain irrotational.

Irrotational hydrodynamics

Irrotational flow: $\nabla_b p_a - \nabla_a p_b = 0 \Leftrightarrow p_a = \nabla_a S$

Hamilton equation: $\partial_t p_a + v^b(\nabla_b p_a - \nabla_a p_b) + \nabla_a H = 0$

Hamilton-Jacobi equation: $\partial_t S + H = 0$

Example: In the dust limit on a Minkowsky background, one obtains a relativistic Burgers equation:

$$\partial_t (v_a / \sqrt{1 - v^2}) + \partial_a (\sqrt{1 + v^2}) = 0 \Leftrightarrow \partial_t S + \sqrt{1 + (\nabla S)^2} = 0$$

Obtained noncovariantly by LeFloch, Makhlofand and Okutmustur, SINUM 50, 2136 (2012) by algebraic manipulation of the Euler equation in Minkowski and Schwarzschild charts. The fact that these are Hamilton equations and can be obtained covariantly for arbitrary spacetimes is unnoticed.

Solutions to HJ equation are NOT unique.
Nevertheless, 'viscosity' solutions to HJ equation are unique.

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USER'S GUIDE TO VISCOSITY SOLUTIONS OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

MICHAEL G. CRANDALL, HITOSHI ISHII, AND PIERRE-LOUIS LIONS

ABSTRACT. The notion of viscosity solutions of scalar fully nonlinear partial differential equations of second order provides a framework in which startling comparison and uniqueness theorems, existence theorems, and theorems about continuous dependence may now be proved by very efficient and striking arguments. The range of important applications of these results is enormous. This article is a self-contained exposition of the basic theory of viscosity solutions.

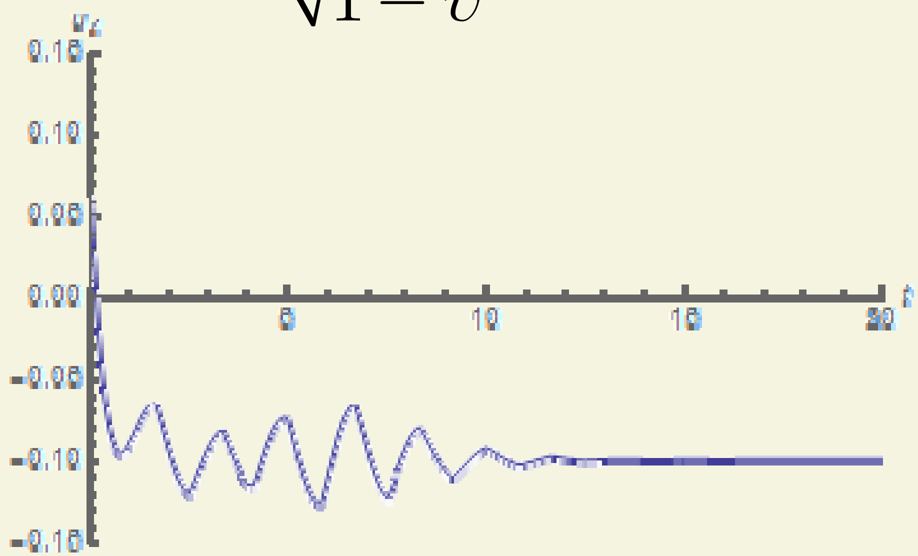
Irrotational hydrodynamics

$$\partial_t S + \sqrt{1 + (\nabla S)^2} = 0$$

Analytic 1+1 solution for homogeneously translating flow:

$$v_x(t, x) = v \Leftrightarrow S(t, x) = \frac{1}{\sqrt{1 - v^2}}(-t + vx)$$

Numerical solution:



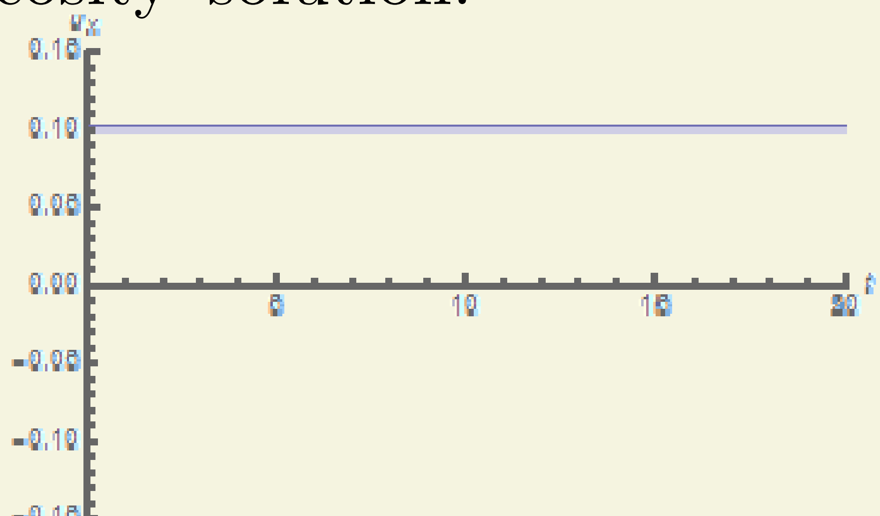
Irrotational hydrodynamics

$$\partial_t S + \sqrt{1 + (\partial_x S)^2} = \varepsilon \partial_x^2 S$$

Analytic 1+1 solution for homogeneously translating flow:

$$\upsilon_x(t,x) = \upsilon \Leftrightarrow S(t,x) = \frac{1}{\sqrt{1-\upsilon^2}}(-t + \upsilon x)$$

Numerical 'viscosity' solution:



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Hamilton equation: $\partial_t p_a + v^b(\nabla_b p_a - \nabla_a p_b) + \nabla_a H = 0$

Hamilton-Jacobi equation: $\partial_t S + H = 0$

For barotropic fluids, the above equation is coupled to the continuity equation, resulting in a system

$$\partial_t \begin{pmatrix} \rho_\star \\ p_i \end{pmatrix} + \partial_k \begin{pmatrix} \rho_\star v^k \\ \delta_i^k H \end{pmatrix} = 0$$

where $\rho_\star := \sqrt{-g}\,\rho u^t = \alpha\sqrt{\gamma}\,\rho u^t, \quad \gamma = \det(\gamma_{ij})$

Characteristics : $\lambda_{1,2}^k = 0$

$$\lambda_{3,4}^k = \alpha(1-\nu^2 c_s^2)^{-1}\{\nu^k(1-c_s^2) \pm c_s(1-\nu^2)^{1/2}[(1-\nu^2 c_s^2)\gamma^{kk} - (1-c_s^2)(\nu^k)^2]^{1/2}\} - \beta^k$$

Complete eigenbasis → The system is strongly hyperbolic (for finite c_s)

Conclusions

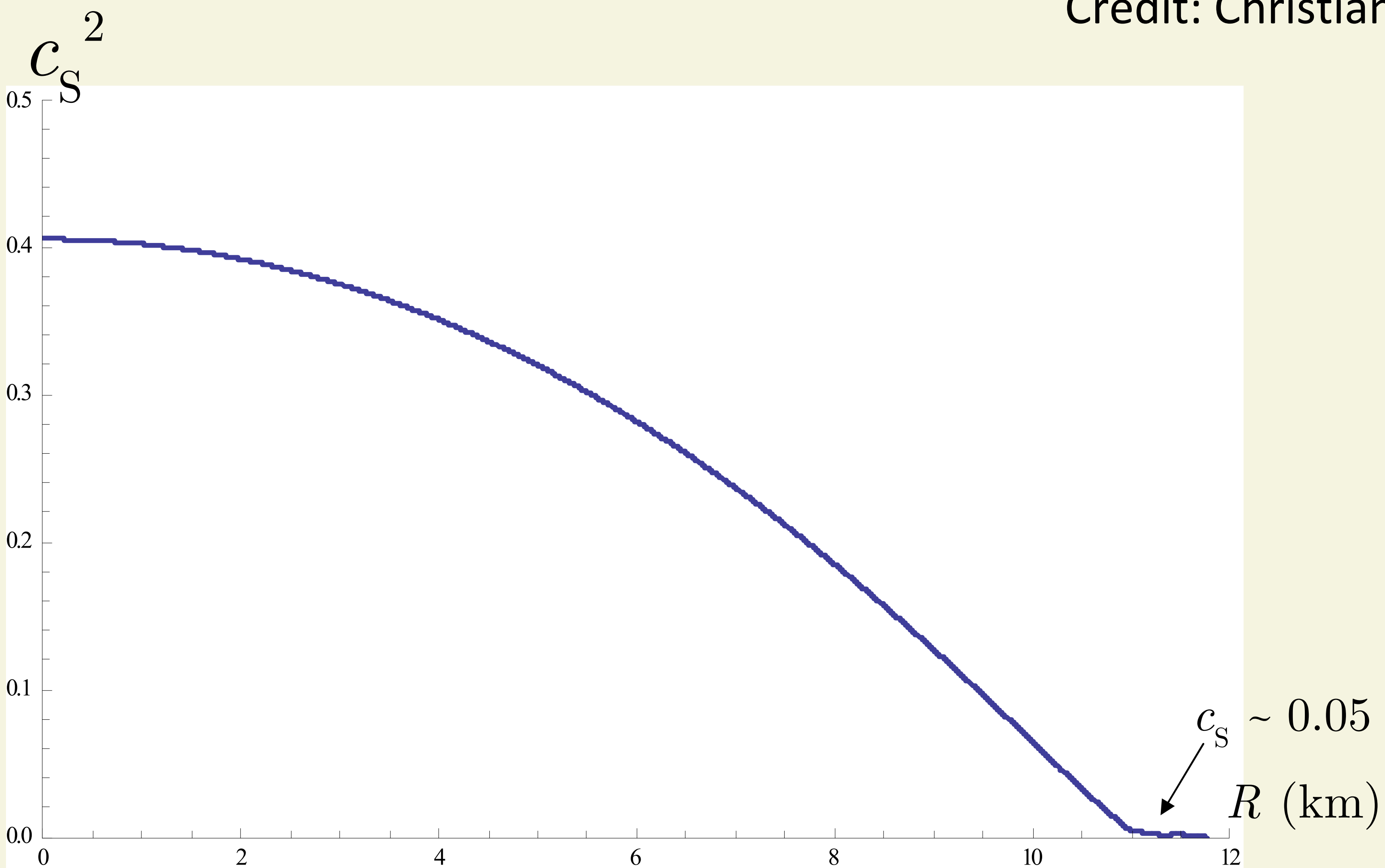
$$\partial_t \begin{pmatrix} \rho_\star \\ p_i \end{pmatrix} + \partial_k \begin{pmatrix} \rho_\star \upsilon^k \\ \delta_i^k H \end{pmatrix} = 0$$

Notable features:

- *Unlike* Valencia, recovery of primitives from conservatives requires no atmosphere: u_i is recovered via dividing $p_i = hu_i$ by specific enthalpy h which is 1 at the surface (no division by zero)
- *Like* Valencia, strong hyperbolicity is lost when $c_s = 0$: eigenbasis not complete, system becomes weakly hyperbolic $\rightarrow$ instability on surface
- Instead of artificial atmosphere, can use crust EOS with small but *nonzero* c_s near surface: sound speed in a realistic NS crust (outer 1 km) $c_s \sim 0.05$

Sound speed profile of a TOV star

Credit: Christian Krüger



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- Instead of artificial atmosphere, can use crust EOS with small but *nonzero* c_s near surface: sound speed in a realistic NS crust (outer 1 km) $c_s \sim 0.05$. Then, extrapolating the EOS to the exterior (h<1) allows one to evolve smooth fields and obtain *pointwise convergence* on the surface, which is unattainable with an artificial atmosphere.

- Scheme may be combined with symplectic integration or constraint damping methods that preserve symplecic structure and circulation
- SPH schemes based on the Lagrangian or Hamiltonian formulation possible
- Extension beyond irrotational flows also possible

Reference

C. Markakis, arXiv:1410.7777