Induced interactions and lattice instability in the inner crust of neutron stars

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In memory of my Friend and Teacher

Vitaliy Bychkov

1968-2015

Work done with

Modes and structure of the crust:

D. N. Kobyakov & C. J. Pethick.

Collective modes of crust – **PRC 87, 055803 (2013);**

Lattice instability – **PRL 112, 112504 (2014);**

Physics of multicomponent superfluid phase

D. N. Kobyakov, L. Samuelsson, E. Lundh, M. Marklund, V. Bychkov & A. Brandenburg.

Quantum hydrodynamics of cold nuclear matter – 2015, unp.

Motivation:
Importance of induced interactions

- Superfluid gaps
- Collective hydro-elastic mode velocities *in the inner crust*
- Collective hydrodynamic mode velocities *in the outer core*
- Lattice structure and role of the neutron liquid *in the inner crust*
- Low-temperature thermal, transport, rotational and magnetic properties

This physics is crucial in the following applications

- ❑ Nuclear structure (especially beyond the neutron drip density)
- ❑ Models of cooling (especially in low-mass x-ray binaries)
- ❑ Models of quasiperiodic oscillations after x-ray flares
- ❑ Modes in magnetars
- ❑ Models of pulsar glitches

Examples of induced interactions in neutron stars

- Neutron-proton coupling renormalizes masses of the Nambu-Goldstone (the Bogoliubov-Anderson) modes both *in the inner crust and in the core*
- Renormalization of the relaxation time of the Cooper instability. (Neutron superfluid gap is reduced by the Gorkov-Melik-Barkhudarov corrections, but amplified by the neutron-phonon interaction *in the inner crust*)
- Proton superfluid gap is strongly influenced by the neutron-induced interactions *in the core*
- Coupling of superfluid neutrons to the magnetic field due to the neutron-proton coupling *in the core*

Plan

1. Induced interactions in physics
2. Dynamic effects of induced neutron-proton interactions in the core
3. Instability of the lattice in the crust
4. Remarks about numerical models of the crust: the shear modulus

Induced interactions in physics

Importance of induced interactions in physics

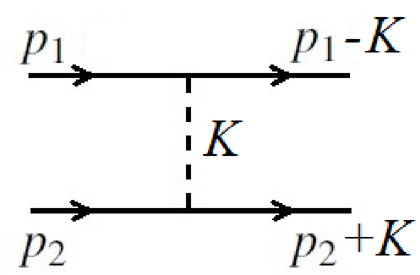
➢ Induced interactions change basic properties of particles

<u>A simple physics example</u>

- Interaction between the electrons in cold metals becomes attractive (electron-phonon induced interactions)

$$\Gamma_{\gamma\delta,\alpha\beta} = \delta_{\alpha\gamma}\delta_{\beta\delta}\left(-\frac{i\omega}{\rho}\right)^2 \frac{\rho k^2}{\omega^2 - u^2k^2 + i0}$$

If for both electrons $|\varepsilon - \varepsilon_F| \ll \omega_D$,
then $\Gamma_{\gamma\delta,\alpha\beta} > 0$ (attraction)



Dynamic effects of neutron-proton induced interactions in the core

Effective field theory

- We need an effective field theory to describe macroscopic phenomena related to superfluidity
- Once the effective degrees of freedom are well defined, the theory may be formulated in terms of these degrees
- The most fundamental principle – the least action principle
- Parameters of such phenomenological theory are chosen so to match the basic properties to properties of the real material

Entrainment in the core

- The Fermi-spheres of neutrons and protons induce kinetic energy contributions $\propto$ terms of 2 order in $\boldsymbol{\nabla}$ and of 4 order in ψ
- **Superfluid entrainment** found in the literature :

$$
\begin{aligned}
&+\lambda_1 (\mathscr{D}\psi_1)(\nabla\psi_2^*)\,\psi_1^*\psi_2 + \lambda_2 (\mathscr{D}^*\psi_1^*)(\nabla\psi_2)\,\psi_1\psi_2^* \\
&+\lambda_3 (\mathscr{D}\psi_1)(\nabla\psi_2)\,\psi_1^*\psi_2^* + \lambda_4 (\mathscr{D}^*\psi_1^*)(\nabla\psi_2^*)\,\psi_1\psi_2.
\end{aligned}
\tag{A1}
$$

- Since $|\psi|^2$ *has a meaning of* superfluid number density, and since the entrainment *must* be Galilean-invariant, therefore Eq. (A1) misses few terms (of the form $\sim |\psi_1|^2|\boldsymbol{\nabla}\psi_2|^2$)
- This little detail is crucial for formulation of the effective field theory of a superfluid mixture

Effective field theory of superfluid-superconductor mixture

- Total energy of superfluid mixture and electromagnetic field (Kobyakov, Samuelsson, Lundh, Marklund, Bychkov & Brandenburg 2015):

$$E_s[\psi_p, \psi_n, \psi_p^*, \psi_n^*, \mathbf{A}] = -\sum\nolimits_{a,b=0\ldots3} \left[\frac{1}{16\pi}(\partial_a A_b - \partial_b A_a)^2\right]$$

$$+\frac{\hbar^2}{2m}\left|\mathscr{D}\psi_p\right|^2 + \frac{\hbar^2}{2m}\left|\nabla\psi_n\right|^2 - \sum\nolimits_{\alpha=p,n} a_\alpha \left|\psi_\alpha\right|^2 + C\left[|\psi_p|, |\psi_n|\right]$$

$$-\frac{1}{2}\rho_{np}\left(\frac{\hbar}{2im}\nabla \ln\frac{\psi_p\psi_n^*}{\psi_p^*\psi_n} - \frac{2e}{mc}\mathbf{A}\right)^2,$$

- «Entrainment parameter» - non-diagonal element of Andreev-Bashkin matrix of superfluid densities (Chamel & Haensel 2006)

$$\rho_{np} = \tilde{\alpha}_{np} n_p n_n.$$

Dynamic effects of induced neutron-proton interactions

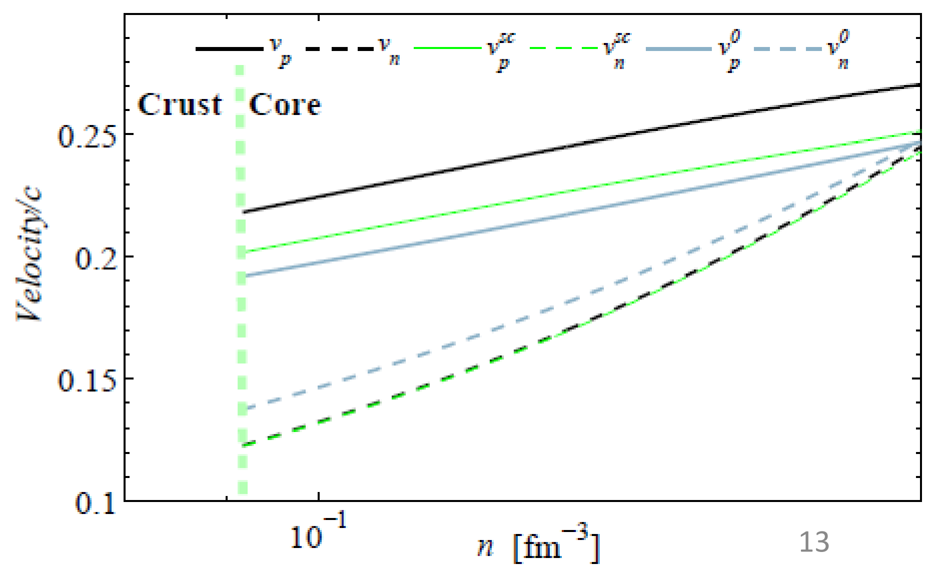
- The fractional quantum of magnetic flux (Alpar, Langer & Sauls 1984; Kobyakov *et al.* 2015):

$$\Phi_{q_p,q_n} = \frac{\pi\hbar c}{e}\left(q_p + \frac{\rho^0_{np}}{\rho^0_{pp}} q_n\right)$$

- Relaxation of relative motion of the electrons and the core of neutron vortices (Alpar, Langer, Sauls 1984)

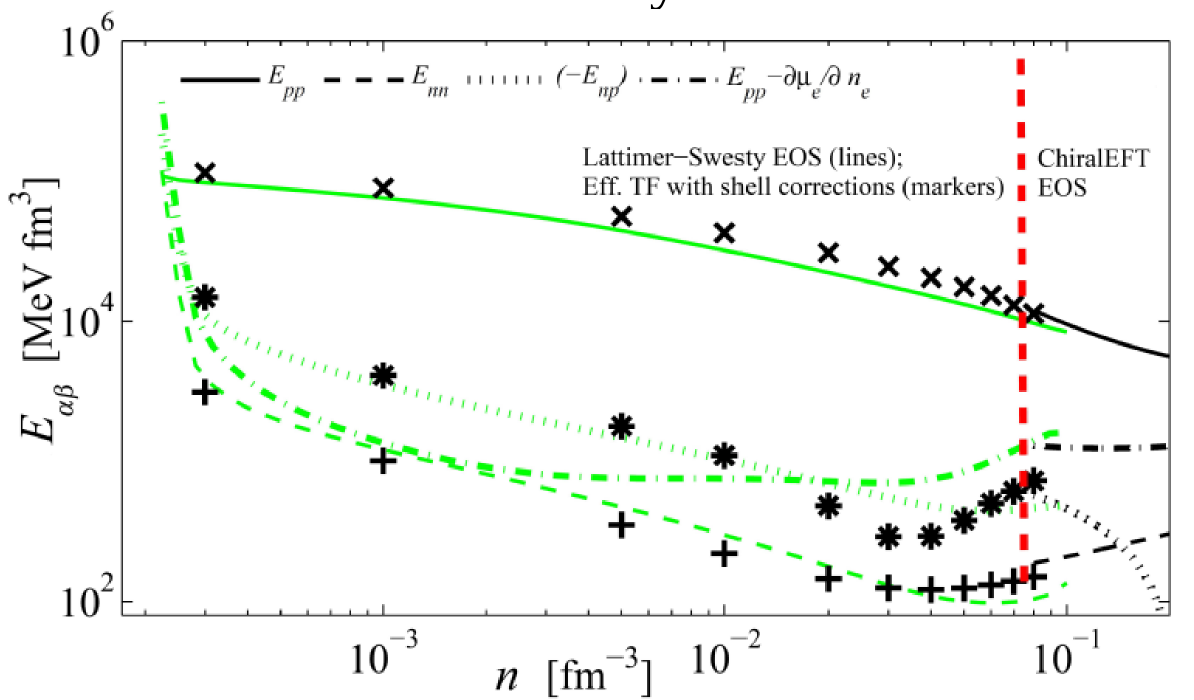
$$\tau_{relax} \sim 1\ [\mathrm{sec}]$$

- Renormalization of masses of the Nambu-Goldstone boson, or sound speeds (Kobyakov *et al.* 2015)



Equation of state of nuclear matter for calculation of $\frac{\partial^2 E}{\partial n_\alpha \partial n_\beta}$

- EOS of uniform nuclear matter based on chiral effective field theory and observations of neutron stars (Hebeler, Lattimer, Pethick & Schwenk 2013)
- Check the behaviour at low densities (Kobyakov *et al.* 2015): matching to the Lattimer-Swesty EOS (Kobyakov & Pethick 2013) and the effective Thomas-Fermi theory with shell corrections (Chamel 2013)



First problem: dissipation

- Dissipation on a simple example

$$i\partial_t \psi(\mathbf{r}) = -\nabla^2 \psi(\mathbf{r}) + g|\psi(\mathbf{r})|^2 \psi(\mathbf{r})$$

$$i\partial_t \psi(\mathbf{k}) = k^2 \psi(\mathbf{k}) + \frac{g}{V^2} \sum_{\mathbf{k}_1,\mathbf{k}_2} \psi(\mathbf{k_1}) \psi(\mathbf{k_2})^* \psi(\mathbf{k} - \mathbf{k}_1 + \mathbf{k}_2)$$

- **Solution:** Pitaevskii (1958), Tsubota, Kamamatsu & Ueda (2002), Kobayashi & Tsubota (2005):

$$i \rightarrow [i - \gamma(k)]$$

- Nuclear superfluids, Kobyakov et al. 2015 for $T = 0+$:

$$[i - \gamma_p(k)]\, \partial_t \psi_p = \ \ldots$$

$$[i - \gamma_n(k)]\, \partial_t \psi_n = \ \ldots$$

Second problem: vortex structure

- *Problem:* Vortex core is too small, if we require the non-linear Schrödinger equation to give correct sound velocities
- In other words: the Nambu-Goldstone boson is too heavy
- **Our solution: *renormalize the NG boson mass spectrally* -** make it small for Fourier harmonics describing the core structure

$$g \to g(k)$$

$$g(k) = g_0 - (g_0 - g_\Delta)\theta(k-1)$$

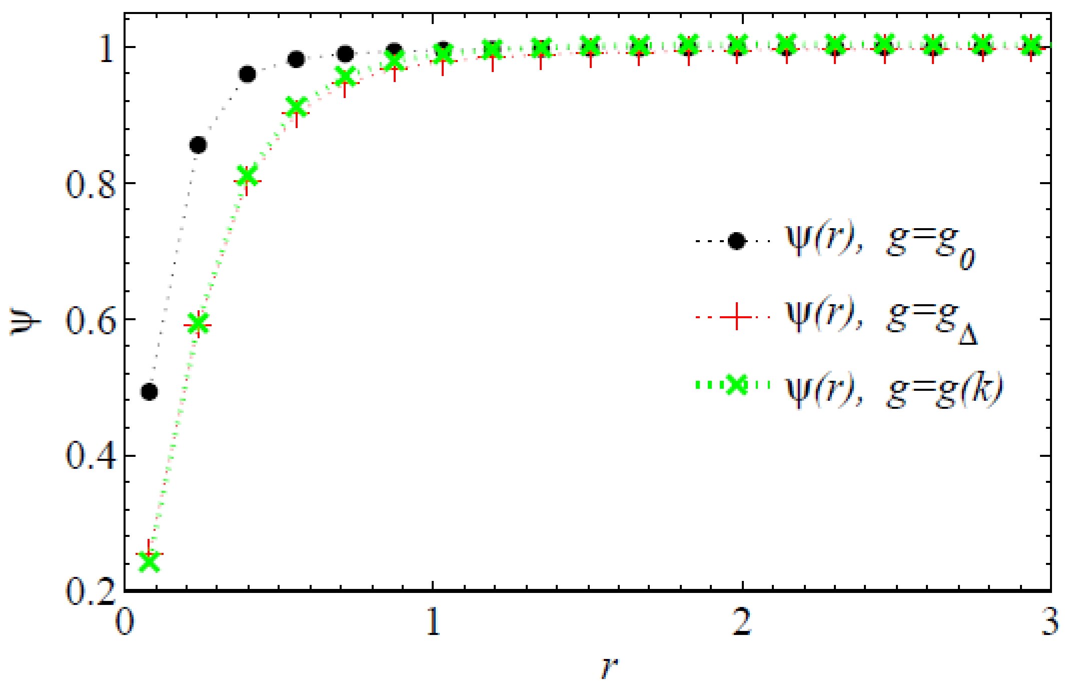
$$\frac{g}{V^2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\psi(\mathbf{k_1})\psi(\mathbf{k_2})^*\psi(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2) \to$$

$$\frac{g(k)}{V^2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\psi(\mathbf{k_1})\psi(\mathbf{k_2})^*\psi(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2)$$

Increasing the core size by the NG-boson renormalization: numerical evidence

- We solve the equations for a single vortex numerically, using the steepest descent method

$$g(k) = g_0 - (g_0 - g_\Delta)\theta(k-1)$$

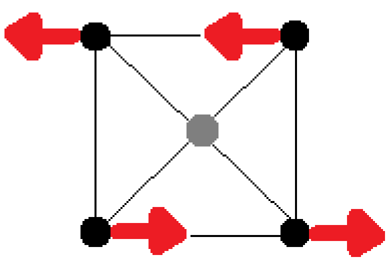


Instability of the lattice in the crust

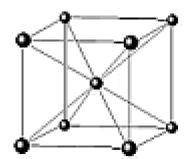
Theoretical description of the lattice

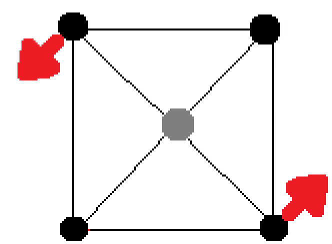
- Elastic energy: $\delta^2 F = \frac{1}{2} \sum_{i,j,k,l} C_{ijkl}\, u_{ij} u_{kl}$
- *Cubic crystal*:

$$\delta^2 F = \frac{1}{2} C_{11}(u_{11}^2 + u_{22}^2 + u_{33}^2) + C_{12}(u_{11}u_{22} + u_{11}u_{33} + u_{22}u_{33}) + 2C_{44}(u_{12}^2 + u_{13}^2 + u_{23}^2)$$



c_{44}





$c' = \frac{C_{11} - C_{12}}{2}$

- Stiffness of the crust material is very anisotropic (next slide)

- *Isotropic solid* (crystallites are small and oriented randomly):

$$\delta^2 F = \frac{1}{2} \sum_{i,j,k,l} \left[K_{\mathrm{eff}} \delta_{ij} \delta_{kl} + \mu_{\mathrm{eff}} \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3} \delta_{ij}\delta_{kl} \right) \right] u_{ij} u_{kl}$$

Elastic anisotropy of crystals

- Zener ratio A $= c_{44}/c'$ measures anisotropy in cubic crystals (A=1 – isotropic)

Crystal	A (Zener ratio)
Silver chloride	0.52
Aluminium	1.22
Silver	3.01
Lithium	8.52

- Coulomb crystal (Fuchs, 1936): $c_{44}/c' \approx 7.4$
- δ-Plutonium: $c_{44}/c' \approx 7.3$

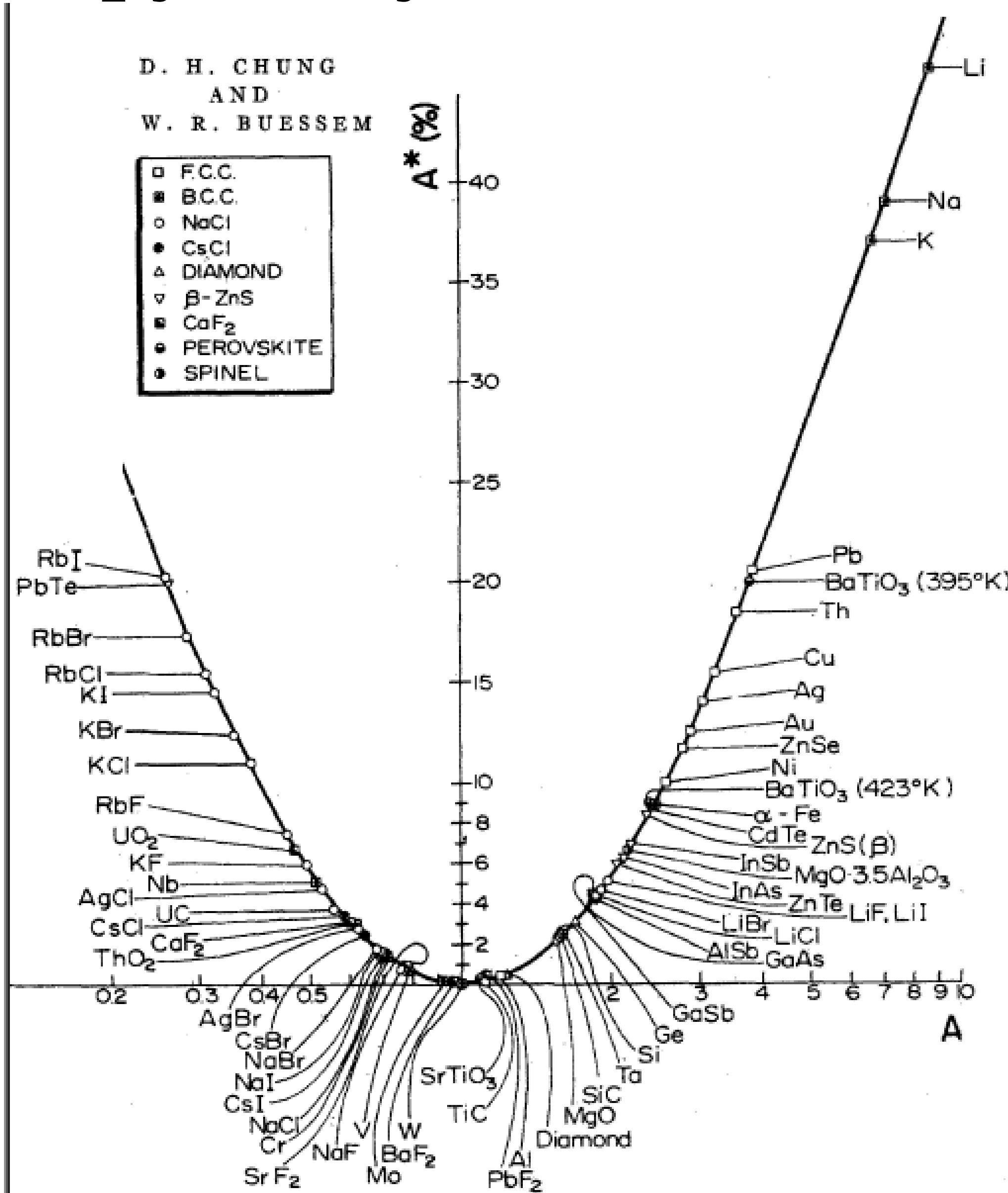


FIG. 1. Graphical representation of the elastic anisotropy for cubic crystals.

Isotropic model of the inner crust

- Continuity equations (linearized)

$$\partial_t \delta n_p + n_p \boldsymbol{\nabla} \boldsymbol{v}_p = 0,$$

$$\partial_t \delta n_n + n_n^s \boldsymbol{\nabla} \boldsymbol{v}_n + n_n^n \boldsymbol{\nabla} \boldsymbol{v}_p = 0.$$

- Euler equations (assuming that solid is isotropic)

$$m \partial_t \boldsymbol{v}_n + \boldsymbol{\nabla}\left(E_{nn} \delta n_n + E_{np} \delta n_p\right) = 0,$$

$$m\left(n_p + n_n^n\right) \partial_t \boldsymbol{v}_p + n_n^n \boldsymbol{\nabla} \mu_n + n_p \boldsymbol{\nabla}\left(\mu_p + \mu_e\right) -$$

Isotropic (pressure)

$$-\frac{4}{3} S\, \boldsymbol{\nabla}(\operatorname{div} \boldsymbol{u}) + S\, \boldsymbol{\nabla} \times (\operatorname{rot} \boldsymbol{u}) = 0,$$

Shear

$$\operatorname{div} \boldsymbol{u} = -\frac{\delta n_p}{n_p}.$$

Induced interactions in the inner crust

- Stability condition is $\delta^2 E > 0$, where

$$\delta^2 E = \frac{1}{2} g_{pp}\, \delta n_p^2 + g_{np} \delta n_p \delta n_n + \frac{1}{2} g_{nn} \delta n_n^2$$

$$\Delta_g = g_{pp} g_{nn} - g_{np}^2$$

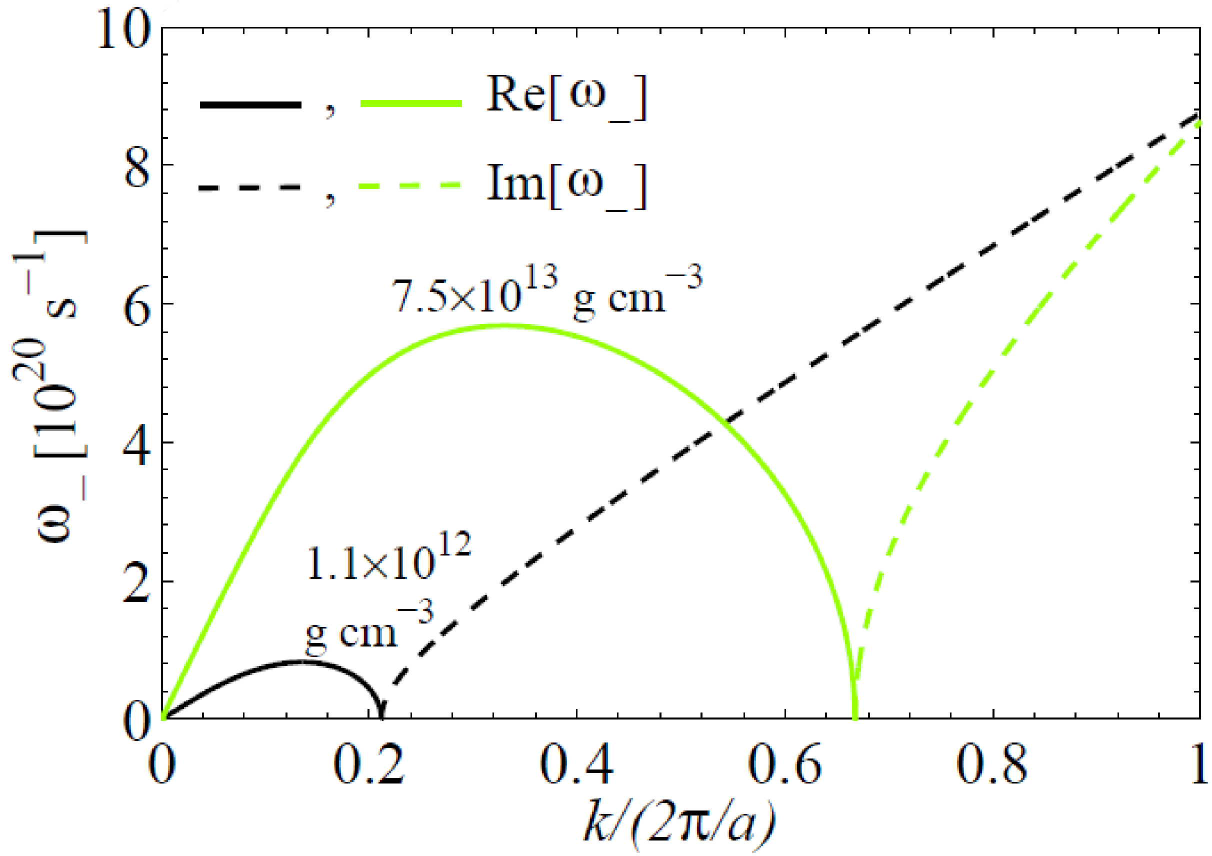
$$\delta^2 E = \frac{1}{2} \frac{\Delta_g}{g_{nn}}\, \delta n_p^2 + \frac{1}{2} g_{nn} \left(\delta n_n + \frac{g_{np}}{g_{nn}}\, \delta n_p \right)^2$$

- Equivalently: $E_{nn} > 0$, $E_{pp} - \dfrac{E_{np}^2}{E_{nn}} = \left.\dfrac{\partial \mu_p}{\partial n_p}\right|_{\mu_n} > 0.$

- Long-wavelength perturbations are stable, since electrons provide a large positive contribution to the effective proton-proton interaction.
- Effective proton-proton interaction is modified by the screening corrections:

$$E_{pp}(k) = E_{pp} - \frac{\partial \mu_e}{\partial n_e} \frac{k^2}{k_{\mathrm{FT}}^2 + k^2}$$

Dispersion relation (with screening corrections) for the in-phase mode



This suggests: The most unstable mode lies at the edge of the 1st Brillouin zone. Now we need to find direction of that mode.

Anisotropic model of the inner crust

- Crystal has cubic symmetry, and the elastic properties are anisotropic.
- Deformation vector field $\boldsymbol{u}(\boldsymbol{r})$.
- Deformation tensor field $u_{ij} = \partial u_i(\boldsymbol{r})/\partial r_j$.
- Energy of deformation of a cubic crystal to the 2nd order ($C_{ijkl} \equiv \lambda_{ijkl}$):

$$\delta F = \frac{1}{2}\lambda_{1111}(u_{11}^2 + u_{22}^2 + u_{33}^2) + \lambda_{1122}(u_{11}u_{22} + u_{11}u_{33} + u_{22}u_{33}) + 2\lambda_{1212}(u_{12}^2 + u_{13}^2 + u_{23}^2)$$

Stability of crystal

- General stability condition: positive definite dynamic matrix

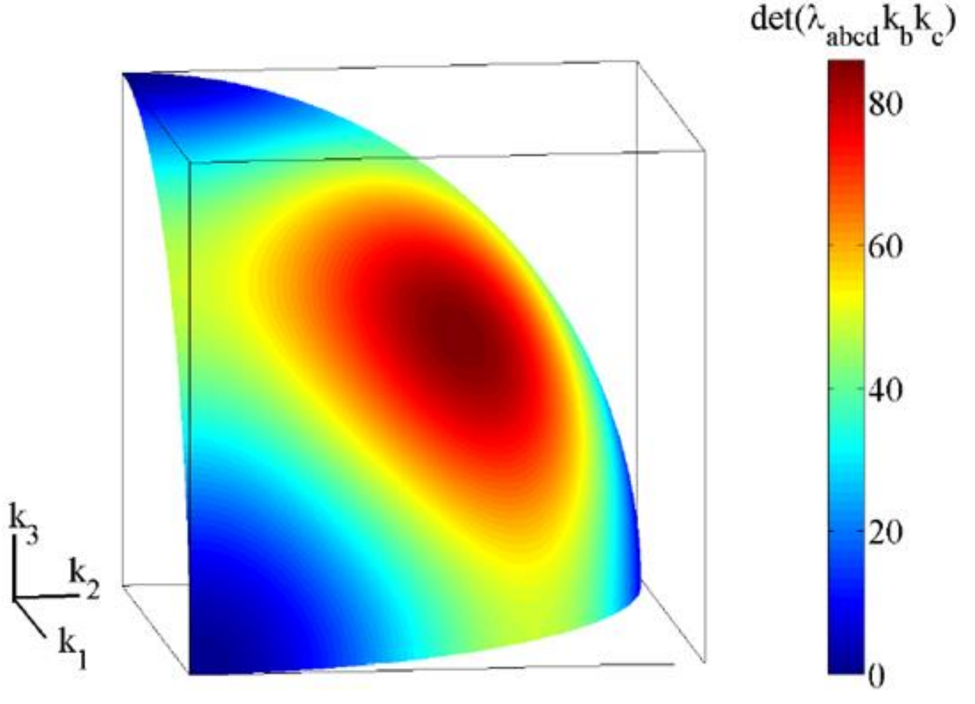
$$\det\left(\sum_{b,c=1,2,3} \lambda_{abcd} k_b k_c\right) > 0$$

- Minimizing the *Gibbs free energy* of a crystal under external pressure is convenient because λ's retain the Voigt symmetry:

$$\lambda_{1212} = \lambda_{1221}$$

The most unstable direction

- Keeping constant the shear modulus $S = (C_{11} - C_{12})/2$ and the coefficient $C_{44} = \lambda_{1212}$, we decrease the bulk modulus $B = (C_{11} + 2C_{12})/3$ and find:



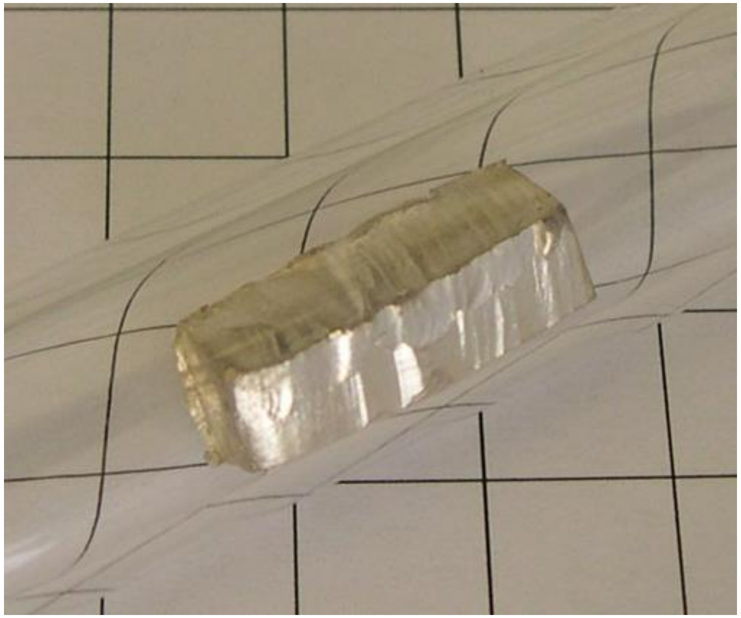
- This result was obtained analytically for $C_{11} - C_{12} > 2C_{44}$ by J. Cahn, *Acta Metallurgica* **10**, 179 (1962).

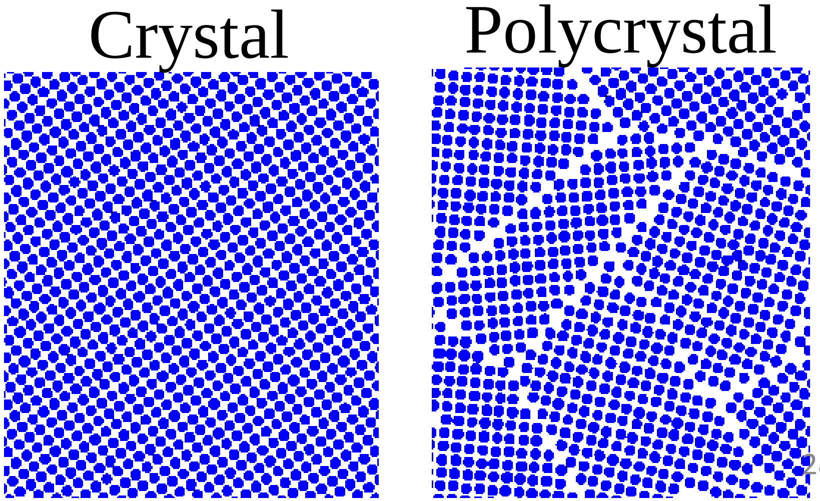
Remarks about numerical models of the crust: The shear modulus (3 slides)

Single crystals and polycrystals

- Hallite (NaCl): cubic

- Lithium: cubic





Averaging the crystalline orientations

- The task is to express the effective moduli of isotropic polycrystalline medium via moduli of pure crystal
- Assume the medium is composed of randomly oriented crystallites, and average the Hooke's law

$$\hat{\sigma} = \hat{\hat{C}}\hat{u} \quad \Leftrightarrow \quad \hat{\hat{C}}^{-1}\hat{\sigma} = \hat{u}$$

$$\langle\hat{\sigma}\rangle = \left\langle\hat{\hat{C}}\hat{u}\right\rangle \qquad \text{or} \qquad \left\langle\hat{\hat{C}}^{-1}\hat{\sigma}\right\rangle = \langle\hat{u}\rangle$$

- **Reminder:**

Deformation tensor (strain) *Stress tensor* *Stiffness tensor*

$$u_{ij} = \frac{1}{2}\left[\partial_i u_j + \partial_j u_i + (\partial_i u_k)\left(\partial_k u_j\right)\right]; \qquad \sigma = \frac{\delta F}{\delta u}; \qquad \sigma_{ij} = C_{ijkl}u_{kl};$$

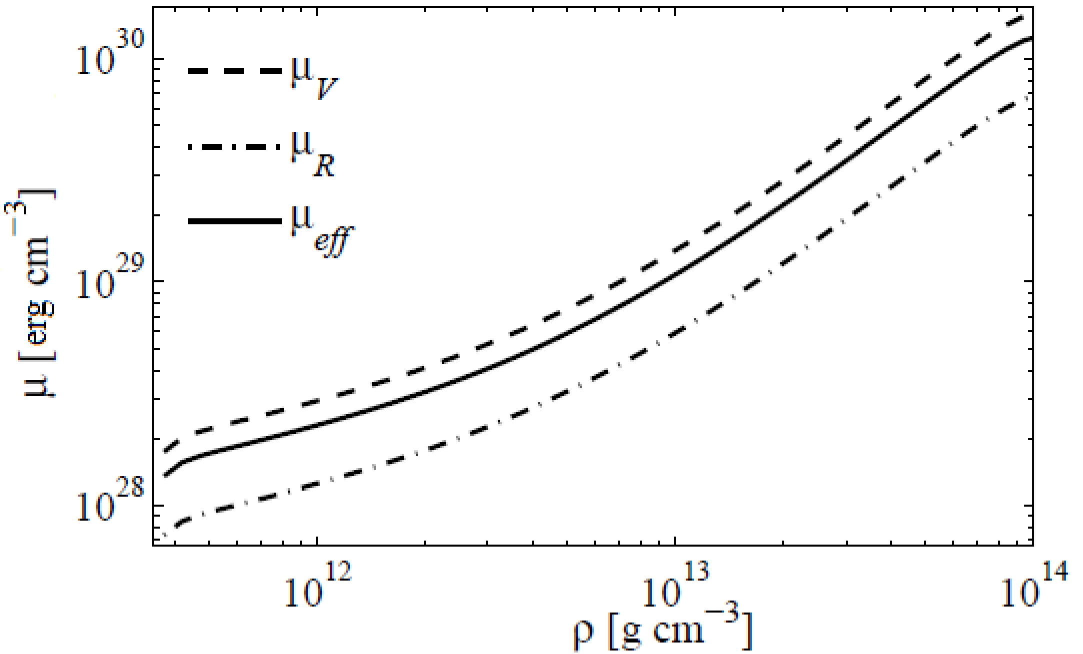
Energy perturbation

$$\delta^2 F = \frac{1}{2}\hat{u}\cdot\hat{\hat{C}}\hat{u} = \frac{1}{2}\hat{u}\cdot\hat{\sigma}$$

Our results for the shear modulus

- Self-consistent model works extremely well in the laboratory.
- For dense polycrystalline matter K is much larger than c' and c_{44}, and we obtain $\mu_{eff} = \frac{c_{44}}{6}\left(1 + \sqrt{1 + 24c'/c_{44}}\right)$.
- Elastic moduli of polycrystalline high-density Coulomb crystal (*neutron star inner crust*) in units $\frac{n_i Z^2 e^2}{a} \sim 10^{30}$ [erg cm^{-3}]

$c' = 0.0997$
$c_{44} = 0.7424$
$\mu_V = 0.4852$
$\mu_R = 0.2071$
$\mu_{\mathrm{eff}} = 0.3462$



Conclusions

- Induced interactions are important
- We calculate collective modes in the outer core using the effective field theory of a superfluid mixture
- We introduce phenomenological *spectral renormalization of the Nambu-Goldstone boson mass* to deal with small length scales
- …and phenomenological dissipation to deal with dynamics, in analogy with terrestrial superfluids and superconductors for $T = 0+$
- We calculate collective modes in the inner crust
- At short wavelength the induced interactions render the lattice unstable
- We find direction of the most unstable mode, which signals a *structural phase transition* in the lattice

Open questions

- What is the equilibrium structure of the lattice?
- How big are crystal domains – the crystallites?
- What is the superfluid neutron density in the inner crust as function of wavenumber?
- What are the phenomenological damping parameters numerically?
- What are the NG boson mass renormalization parameters numerically?